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A STUDY OF HELICALLY REINFORCED CYLINDERS UNDER AXIALLY SYMMETRIC LOADS AND APPLICATION TO STRAND MATHEMATICAL MODELLING

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Abstract—Locally transversely isotropic elastic material can be used in a cylinder configuration in which the material principal direction is helically wound. Stress–strain equations are obtained for such a cylinder under axially symmetric loading: axial force, twisting moment and uniform radial internal and external pressure forces. The equations are generalized to a system of n coaxial tubes with an isotropic core. Finally, it is shown how this can be used to represent the global mechanical behaviour of strand-like systems such as overhead electrical conductors and cables.

INTRODUCTION

Electrical power is often transmitted through a conducting metal core surrounded by a number of insulating layers. The latter may be fibre reinforced and the mechanical behaviour of the resulting conductor can be studied as a system of coaxial locally, transversely isotropic tubes with fibres helically wound in each cylinder. Generally, the winding direction alternates from one layer to next. On the other hand, overhead electrical conductors consist simply of several layers of aluminum wires, which may be wound around a steel core in the case of ACSR (aluminum conductor steel reinforced). Load carrying cables are very similar. Also, complex combinations of the foregoing can be found for example in underwater power cables, in which load carrying steel cables are embedded in an insulating matrix surrounding conducting copper wires. Such systems have been described by Carlson *et al.* (1973).

The mechanical behaviour of stranded systems such as cables and overhead electrical conductors is usually studied using discrete models, in which each wire is considered separately. This is of course advantageous in some problems, such as structural integrity assessment. Various assumptions have been made concerning the contact conditions, resulting in several models, mostly for systems under axial load with varying degrees of rotational restraint at the ends. Typical of this approach are works by Hruska (1952), Machida and Durelli (1973), Phillips and Costello (1985), Knapp (1979) and Lanteigne (1985). These models are generally developed for a small number of wires. However, when this number increases, as well as the number of layers, it may be appealing, at a certain point, to resort to a continuous or, at least, semi-continuous model, since a finite number of cylinders is still needed, corresponding to the same number of layers as in the actual system. This idea has already been explored by Raoof (1983). In his work, Raoof replaces each cable layer with an orthotropic lamina. The natural axes of each lamina are wound helically to form a cylinder. Elastic parameters are obtained from interwire contact considerations. However, elasticity plays a role only in the cylinder tangential plane. Radial motion is solely due to the variation of the helix angle under tensile load. The effect of Poisson's ratio in this direction is neglected. In the present analysis, the case of cylinders made of a locally transversely isotropic material under various axially symmetric loads is first studied. A comparison is then made with some discrete models. Under the no-slip assumption, available elastic parameters are adjusted to fit selected discrete model global characteristics.

GOVERNING EQUATIONS

The basic constitutive equations for a transversely isotropic, linearly elastic material the preferred direction of which is that of a unit vector $\{a\}$, have been expressed by Spencer *et al.* (1984) as

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij}+2\mu_T\varepsilon_{ij}+\alpha(a_k a_l\varepsilon_{kl}\delta_{ij}+a_i a_j\varepsilon_{kk})+2(\mu_L-\mu_T)(a_i a_k\varepsilon_{kj}+a_j a_k\varepsilon_{ki})+\beta a_i a_j a_k a_l\varepsilon_{kl} \tag{1}$$

where α, β, μ_L and μ_T are elastic parameters. These parameters are related to the usual elastic moduli in the following fashion:

$$\mu_L = G_L \tag{2}$$

$$\mu_T = G_T = E_T/2(\nu_T+1) \tag{3}$$

$$\lambda = E_T\left(\frac{E_T}{E_L}\nu_T+\nu_L^2\right)\Big/(\nu_T+1)\gamma \tag{4}$$

$$\alpha = E_T\left[\nu_L(\nu_T-\nu_L+1)-\frac{E_T}{E_L}\nu_T\right]\Big/(\nu_T+1)\gamma \tag{5}$$

$$\beta = E_T\frac{1-\nu_T}{\gamma}-4G_L+2G_T-2\alpha-\lambda \tag{6}$$

$$\gamma = \frac{E_T}{E_L}(1-\nu_T)-2\nu_L^2. \tag{7}$$

In these equations, subscript L corresponds to the longitudinal (fibre) direction, while T corresponds to any transverse direction. Thus, E_L and E_T are the elastic moduli in the L- and T-directions, respectively. G_L is the shearing modulus for shear strain between the L-direction and any T-direction while G_T is the shearing modulus for shear strain between any two orthogonal T-directions. ν_T is Poisson's ratio for stress in the T-direction and strain in the perpendicular T-direction. ν'_L corresponds to strain in the T-direction due to stress in the L-direction, while ν_L corresponds to strain in the L-direction under stress in the T-direction. These seven moduli must satisfy the two following conditions:

$$\nu'_L = \frac{E_L}{E_T}\nu_L \tag{8}$$

$$G_T = \frac{E_T}{2(1+\nu_T)} \tag{9}$$

leaving only five independent elastic constants. These constants, to be physically meaningful, have to satisfy several inequality constraints which can be found in any book on composite materials, such as the one by Jones (1975).

The helix angle θ of the fibres in a cylinder is measured with respect to the cylinder axis (Fig. 1). Unit vectors $\{e_1\}$, $\{e_2\}$, $\{e_3\}$ are in the radial, circumferential and axial

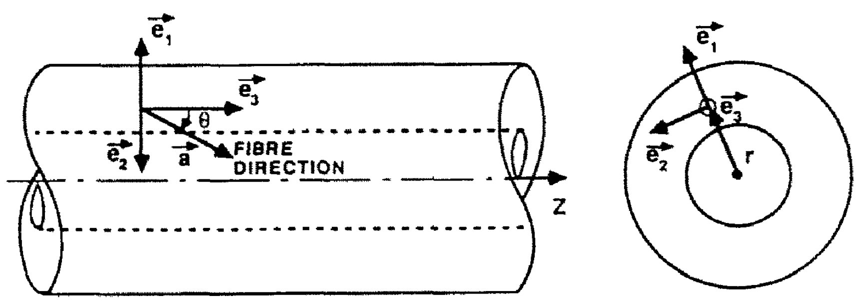

Fig. 1. Locally transversely isotropic cylinder: system of coordinates.

directions, respectively. With respect to this system, the unit vector $\{a\}$ has the following components:

$$a_1 = 0, \qquad a_2 = \sin\theta, \qquad a_3 = \cos\theta. \tag{10}$$

The stress–strain equation, eqn (1), can be written in matrix form as

$$\{\sigma\} = [Q]\{e\} \tag{11}$$

with

$$\{\sigma\}^{\mathrm{T}} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}\} \tag{12}$$

$$\{e\}^{\mathrm{T}} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{31}, 2\varepsilon_{12}\} \tag{13}$$

and $[Q]$ the stiffness matrix defined by eqns (A1)–(A14).

DISPLACEMENTS AND STRESSES FOR ONE CYLINDER

Under an axially symmetric load, following an argument by Verma and Rana (1983), one may assume that a point initially at (r, ψ, z) goes to $(Ur, \psi+\phi z, \omega z)$ after deformation. Thus the displacement components are

$$u_1 = u = (U-1)r \tag{14}$$

$$u_2 = v = \phi z r \tag{15}$$

$$u_3 = w = (\omega-1)z = \varepsilon z \tag{16}$$

where U is a non-dimensional function of r, to be found, ϕ the twist angle per unit length and $\varepsilon = \omega - 1$ the axial strain. Displacements (15) and (16) are based on the assumption that the end of the cylinder, $z = 0$, is fixed. These displacements result in the following strains:

$$\varepsilon_{11} = U - 1 + \frac{\mathrm{d}U}{\mathrm{d}r}r \tag{17}$$

$$\varepsilon_{22} = U - 1 \tag{18}$$

$$\varepsilon_{33} = \omega - 1 = \varepsilon \tag{19}$$

$$\varepsilon_{23} = \tfrac{1}{2}\phi r \tag{20}$$

$$\varepsilon_{13} = \varepsilon_{12} = 0 \tag{21}$$

which satisfy compatibility conditions. Finally, the stress equilibrium equations yield

$$r^2\frac{\mathrm{d}^2U}{\mathrm{d}r^2} + 3r\frac{\mathrm{d}U}{\mathrm{d}r} + U\frac{Q_{11}-Q_{22}}{Q_{11}} + \phi r\frac{2Q_{14}-Q_{24}}{Q_{11}} + \omega\frac{Q_{13}-Q_{23}}{Q_{11}} + \frac{Q_{22}+Q_{23}-Q_{11}-Q_{13}}{Q_{11}} = 0 \tag{22}$$

where Q_{ij} are elements of the material stiffness matrix $[Q]$.

A general solution to this equation is

$$U = q_1 r^{k_1} + q_2 r^{k_2} + a\phi r + b\varepsilon + 1 \tag{23}$$

in which

$$k_1 = -1 + \sqrt{(Q_{22}/Q_{11})} \tag{24}$$

$$k_2 = -1 - \sqrt{(Q_{22}/Q_{11})} \tag{25}$$

$$a = \frac{Q_{24} - 2Q_{14}}{4Q_{11} - Q_{22}} \tag{26}$$

$$b = \frac{Q_{23} - Q_{13}}{Q_{11} - Q_{22}}, \tag{27}$$

q_1, q_2 are undetermined parameters, while ϕ and ω are constants related to the type of end loading being imposed on the cylinder. Stresses can be derived from eqn (23)

$$\sigma_{11} = q_1 C_1 r^{k_1} + q_2 D_1 r^{k_2} + E_1 \phi r + F_1 \varepsilon \tag{28}$$

$$\sigma_{22} = q_1 C_2 r^{k_1} + q_2 D_2 r^{k_2} + E_2 \phi r + F_2 \varepsilon \tag{29}$$

$$\sigma_{33} = q_1 C_3 r^{k_1} + q_2 D_3 r^{k_2} + E_3 \phi r + F_3 \varepsilon \tag{30}$$

$$\sigma_{23} = q_1 C_4 r^{k_1} + q_2 D_4 r^{k_2} + E_4 \phi r + F_4 \varepsilon \tag{31}$$

$$\sigma_{12} = \sigma_{13} = 0. \tag{32}$$

Parameters C_i, D_i, E_i, F_i ($i = 1, 2, 3, 4$) depend on the elastic constants as indicated in eqns (A15)–(A18). The case of a single transversely isotropic cylinder subjected to axial loads (force and moment) together with internal and external pressure is shown in eqns (A28)–(A37).

SEVERAL CYLINDERS WITH AN ISOTROPIC CORE

We consider a system of n concentric cylinders. Cylinder i extends from inner radius r_{i-1} to outer radius r_i. The system external radius is thus r_n. The previous equations apply to each separate cylinder with all the parameters and functions indexed accordingly. For example, displacements become $u_{,i}$ $v_{,i}$ $w_{,i}$ stresses $\sigma_{11,i}$, etc. This does not apply to global coordinates r and z and to global deformation parameters ϕ, ω, ε which are assumed to be the same for all cylinders. That is to say, we assume perfect bonding between cylinders. An elastic isotropic core is also assumed, extending from $r = 0$ to r_0. For the core, eqn (23) simply becomes

$$U_{,0} = q_{1,0} r \tag{33}$$

with the corresponding stress field

$$\sigma_{11,0} = \frac{E_c}{\gamma_c}[q_{1,0} + \nu_c \varepsilon] \tag{34}$$

$$\sigma_{22,0} = \sigma_{11,0} \tag{35}$$

$$\sigma_{33,0} = \frac{E_c}{\gamma_c}[(1-\nu_c)\varepsilon + 2\nu_c q_{1,0}] \tag{36}$$

$$\sigma_{23,0} = G_c \phi r \tag{37}$$

$$\sigma_{31,0} = \sigma_{12,0} = 0 \tag{38}$$

where

$$\gamma_c = (1+\nu_c)(1-2\nu_c).$$

Boundary conditions at each interface r_i are

$$u_{,i}|_{r=r_i} = u_{,i+1}|_{r=r_i} \qquad (i = 0, \ldots, n-1) \tag{39}$$

$$\sigma_{11,i}|_{r=r_i} = \sigma_{11,i+1}|_{r=r_i} \qquad (i = 0, \ldots, n-1) \tag{40}$$

while at the outer radius $r = r_n$, there may be an external pressure p_e such that

$$q_{1,n} C_{1,n} r_n^{k_{1,n}} + q_{2,n} D_{1,n} r^{k_{2,n}} + E_{1,n} \phi r_n + F_{1,n} \varepsilon = -p_e. \tag{41}$$

In matrix notation, these $2n+1$ equations become

$$[M]\{q\} = \phi\{X'\} + \omega\{W'\} + \{Y'\} \tag{42}$$

with vectors $\{X'\}$, $\{W'\}$, $\{Y'\}$ and the matrix $[M]$ defined in eqns (A20)–(A22). Multiplying by $[M]^{-1}$ yields

$$\{q\} = \phi\{X\} + \omega\{W\} + \{Y\}. \tag{43}$$

GLOBAL STIFFNESS MATRIX

The total axial load N on the cylinder system is given by

$$N = \sum_{i=0}^{n} 2\pi \int_{r_{i-1}}^{r_i} \sigma_{33,i} r \, \mathrm{d}r \tag{44}$$

with the convention that $r_{-1} = 0$.

Substitution of axial stress eqn (30) yields

$$N = \omega N_1 + \phi N_2 + N_3. \tag{45}$$

In the same fashion, the twisting moment M on the system is

$$M = \sum_{i=0}^{n} 2\pi \int_{r_{i-1}}^{r_i} \sigma_{23,i} r^2 \, \mathrm{d}r \tag{46}$$

which yields after integration

Table 1. Geometry and elastic properties of a typical 6 × 1 steel strand

Core	Layer
$E_c = 207$ GPa	$E_1 = 207$ GPa
$\nu_c = 0.3$	$\nu_1 = 0.3$
$r_c = 1.27$ mm	$r_1 = 3.81$ mm
	$\theta = 5$
	$m_1 = 6$ wires

$$M = \omega M_1 + \phi M_2 + M_3. \tag{47}$$

Equations (45) and (47) can be written in matrix form as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 \\ M_1 & M_2 \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \phi \end{Bmatrix} + \begin{Bmatrix} N_1 + N_3 \\ M_1 + M_3 \end{Bmatrix} \tag{48}$$

with the matrix coefficients defined in eqns (A51) and (A52). It can be checked that the 2×2 matrix is symmetric with $N_2 = M_1$. The last term of eqn (48) depends only on the applied pressure p_c. It vanishes when $p_c = 0$.

APPLICATION TO A STRAND

A continuous model for such systems will be appropriate when the number of wires in each layer is sufficiently large. However, in order to show the principle of parameter identification and a comparison with available results, we consider a simple, one-layer model consisting of seven identical wires, one of which is the core while the other six are helically wound around it. Parameters are given in Table 1. Such a system has been studied extensively under various assumptions. After linearization, one should get a relationship similar to eqn (48), without the pressure term, between the axial loads N and M, and the global deformation parameters ε and ϕ. Matrix parameters depend on material properties, wire radii and helix angle. However, it has been shown (Blouin, 1988) that most models do not yield a symmetrical matrix. Thus, for these models, after linearization, we obtain

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \phi \end{Bmatrix} \tag{49}$$

in which, generally, $B \neq C$. We calculate A, B, C and D for this seven-wire system from Phillips and Costello (1985). The same coefficients calculated from Machida and Durelli (1973) and Knapp (1979), with a compressible core, differ by about 1% from these values.

Elastic properties of the continuous model, consisting of a transversely isotropic tube with an isotropic core, are selected as follows. Core properties are the same as those of the actual wire material. However, the layer longitudinal modulus $E_{L,1}$ has to be reduced if one decides to keep the same cylinder thickness as that of the six-wire layer. Thus, since each wire has a radius $r_0 = r_c$

$$E_{L,1} = \frac{6\pi r_0^2}{\pi(r_1^2 - r_0^2)} E_c = 0.75 E_c. \tag{50}$$

Here we neglect the slight effect of wire helix inclination. The same helix angle is taken for the tube fibre direction as for the wire strands. The other elastic constants have been chosen as

Table 2. Adjusted stiffness coefficients of transversely isotropic cylinder model

N_1 ($\times 10^6$ N)	N_2 ($\times 10^6$ N mm)	M_1 ($\times 10^6$ N mm)	M_2 ($\times 10^6$ N mm^2)
7.026	1.234	1.234	2.398

Table 3. Closeness of fit between continuous and discrete model coefficients, as calculated from Hruska (1952)

$\|N_1-A\|/A$	$\|N_2-B\|/B$	$\|M_1-C\|/C$	$\|M_2-D\|/D$
3.9%	11.4%	6.7%	8.2%

$$E_{T,1} = E_{L,1}/400 \tag{51}$$

$$\nu'_{L,1} = \nu_{T,1} = 0.3 \tag{52}$$

$$G_{L,1} = 40G_{T,1} \tag{53}$$

where $G_{T,1}$ is obtained from eqn (9). It can be checked that such values are admissible.

Corresponding values of the stiffness coefficients N_1, N_2, M_1, M_2 of eqn (48) are shown in Table 2, while the degree of closeness between these coefficients and those calculated from Phillips and Costello's model is shown in Table 3. Adjustment was made through parameters $E_{T,1}$ and $G_{L,1}$ only and a closer agreement between both models could be sought, for example by varying $\nu'_{L,1}$ and $\nu_{T,1}$ independently. Since the discrete model is already based on a number of assumptions, this has not been deemed necessary. Thus, using the selected elastic constants, some numerical results are shown in Figs 2 and 3 for the case where a zero rotation is imposed on the system. In Fig. 2, the radial stress is shown as a function of r for axial loads $N = 8$ and 16 kN and zero external pressure. In particular, one can obtain the interface pressure between the core and the tube. This pressure reaches 1.17 MPa for an axial load of 8 kN, as compared with a value of 1.51 MPa if calculated by Hruska's formulas for the same geometric parameters (diameters and helix angle) and assuming a uniform distribution of line loads over the core surface. The 30% difference can be considered as very encouraging considering that both approaches are basically different and Hruska's model being a very simplified one. In Fig. 3, shear stress in the cylinder is shown for various combinations of axial load N and external pressure p_e. Since there is no end rotation, the shear stress is obviously zero in the isotropic core.

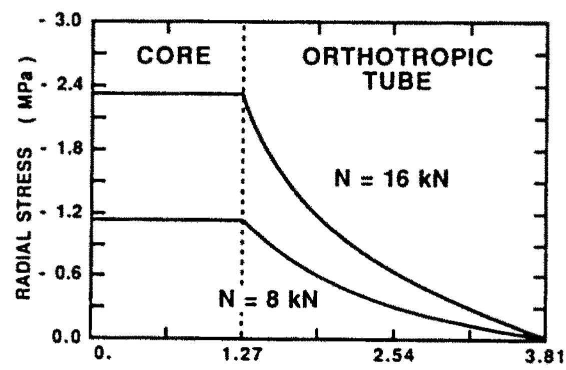

Fig. 2. Radial stress in cylinder and core under axial load and zero twist ($\phi = 0$), with no external pressure.

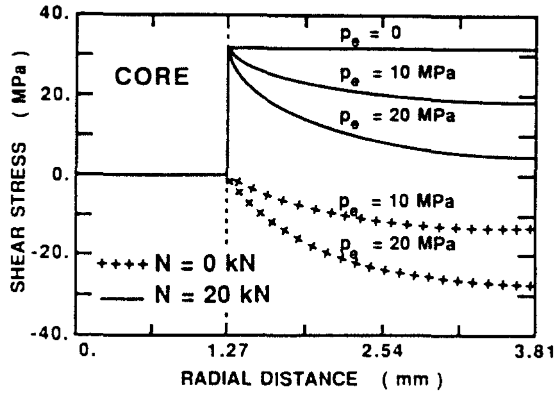

Fig. 3. Shear stress in cylinder and core under axial load and zero twist ($\phi = 0$), with external pressure.

CONCLUSION

The primary objective of this work was to derive equations describing the behaviour of a system of coaxial cylinders under axisymmetric loading. Such a system is mechanically plausible in its own right. However, the longer range objective is the application of such a semi-continuous model to strand-like systems such as electrical conductors and cables. Indeed discrete models are generally not appropriate for a study of the bending behaviour, which is of great importance in such phenomena as transverse vibration and fatigue. A continuous model might shed light on some aspects of this problem. Extension of the present continuous model to the bending problem shall mean abandoning the simple Verma and Rana displacement equations, eqns (14) (16), containing only one unknown function and two unknown parameters. Of course, a simple starting point would be to use the usual Bernoulli Euler hypothesis on plane cross-sections. However, it has been shown by Bauchau (1985) that this hypothesis is inadequate for beams constituted with anisotropic materials. Thus, a more rigorous approach should take into account possible cross-section warping. Finally, a crucial step is in the proper selection of the elastic constants. The procedure which has been used here, based on the adjustment of these constants to a known theoretical discrete model, was only for illustrative purposes. A more rational approach would be, following McConnell and Zemke (1982), to use experimental data, for a given system, or else, to go deeper into contact considerations such as those used by Raoof (1983). Conversely, the present method could be used to compare the consistency of various discrete models and to evaluate the influence of several of the assumptions which are often made.

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REFERENCES

Bauchau, O. A. (1985). A beam theory for anisotropic materials. *J. Appl. Mech.* **52**, 416.

Blouin, F. (1988). Modélisation des solides toronnés à l'aide de la théorie des matériaux élastiques orthotropes transversalement isotropes. M.Sc. Thesis, Laval University, Quebec, Canada.

Carlson, A. D., Kasper, R. G. and Tucchio, M. A. (1973) A structural analysis of a multiconductor cable. *Naval Underwater Systems Center Technical Report* 4549, Newport, Rhode Island.

Hruska, F. H. (1952). Radial forces in wire ropes. *Wire Wire Prod.* **27**(5), 459.

Jones, R. M. (1975). *Mechanics of Composite Materials*. McGraw-Hill, New York.

Knapp, R. H. (1979) Derivation of a new stiffness matrix for helically armored cables considering tension and torsion. *Int. J. Numer. Meth. Engng* **14**, 515.

Lanteigne, J. (1985). Theoretical estimation of the response of helically armored cables to tension, torsion and bending. *J. Appl. Mech.* **52**, 423.

Machida, S. and Durelli, A. J. (1973). Response of a strand to axial and torsional displacements. *J. Mech. Engng Sci.* **15**(4), 241.

McConnell, K. G. and Zemke, W. P. (1982). A model to predict the coupled axial torsion properties of ACSR conductors. *Expl. Mech.* **22**, 237.

Phillips, J. W. and Costello, G. A. (1985). Analysis of wire ropes with internal wire-rope cores. *J. Appl. Mech.* **52**, 510.

Raoof, M. (1983). Interwire contact forces and the static, hysteretic and fatigue properties of multi-layer structural strands. Ph.D. Thesis, Imperial College, London.

Spencer, A. J. M., Rogers, T. G., England, A. H., Pipkin, A. C. and Parker, D. F. (1984). Continuum theory of the mechanics of fiber-reinforced composites. *CISM Courses and Lectures No.* 282, International Centre for Mechanical Sciences, Udine, Italy.

Verma, P. D. S. and Rana, O. H. (1983). Rotation of a circular cylindrical tube reinforced by fibres lying along helices. *Mech. Mater.* **2**, 353.

APPENDIX

1. *Material stiffness matrix* $[Q]$

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & Q_{34} & 0 & 0 \\ Q_{14} & Q_{24} & Q_{34} & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & Q_{56} \\ 0 & 0 & 0 & 0 & Q_{56} & Q_{66} \end{bmatrix}. \tag{A1}$$

Letting $c = \cos\theta$ and $s = \sin\theta$, with helix angle θ, the matrix coefficients are

$$Q_{11} = \lambda + 2\mu_T \tag{A2}$$

$$Q_{12} = \lambda + \alpha s^2 \tag{A3}$$

$$Q_{13} = \lambda + \alpha c^2 \tag{A4}$$

$$Q_{14} = \alpha sc \tag{A5}$$

$$Q_{22} = \lambda + 2\mu_T + 2s^2[\alpha + 2(\mu_L - \mu_T)] + \beta s^4 \tag{A6}$$

$$Q_{23} = \lambda + \alpha + \beta s^2 c^2 \tag{A7}$$

$$Q_{24} = [\alpha + 2(\mu_L - \mu_T) + \beta s^2] sc \tag{A8}$$

$$Q_{33} = \lambda + 2\mu_T + 2c^2[\alpha + 2(\mu_L - \mu_T)] + \beta c^4 \tag{A9}$$

$$Q_{34} = [\alpha + 2(\mu_L - \mu_T) + \beta c^2] sc \tag{A10}$$

$$Q_{44} = \mu_L + \beta s^2 c^2 \tag{A11}$$

$$Q_{55} = \mu_T s^2 + \mu_L c^2 \tag{A12}$$

$$Q_{56} = (\mu_L - \mu_T) sc \tag{A13}$$

$$Q_{66} = \mu_T c^2 + \mu_L s^2. \tag{A14}$$

2. *Stress equations constants*

For $i = 1, 2, 3, 4$ and taking into account the symmetry of matrix $[Q]$

$$C_i = Q_{1i}(k_1 + 1) + Q_{2i} \tag{A15}$$

$$D_i = Q_{1i}(k_2 + 1) + Q_{2i} \tag{A16}$$

$$E_i = Q_{4i} + a(2Q_{1i} + Q_{2i}) \tag{A17}$$

$$F_i = Q_{3i} + b(Q_{1i} + Q_{2i}). \tag{A18}$$

3. *Multi-layer case with isotropic core*

$$\{q\}^{T} = \{q_{1,0}q_{1,1}q_{2,1}\cdots q_{1,i}q_{2,i}\cdots q_{1,n}q_{2,n}\} \tag{A19}$$

$$\{X'\} = \begin{Bmatrix} a_{,1}r_0 \\ E_{1,1}r_0 \\ (a_{,2}-a_{,1})r_1 \\ (E_{1,2}-E_{1,1})r_1 \\ \text{-----------} \\ (a_{,i+1}-a_{,i})r_i \\ (E_{1,i+1}-E_{1,i})r_i \\ \text{-----------} \\ (E_{1,n}-E_{1,n-1})r_{n-1} \\ -E_{1,n}r_n \end{Bmatrix} \tag{A20}$$

$$\{W'\} = \begin{Bmatrix} b_{,1} \\ E_{1,1}-E_c\nu_c/\gamma_c \\ b_{,2}-b_{,1} \\ F_{1,2}-F_{1,1} \\ \text{-----------} \\ b_{,i+1}-b_{,i} \\ F_{1,i+1}-F_{1,i} \\ \text{-----------} \\ F_{1,n}-F_{1,n-1} \\ -F_{1,n} \end{Bmatrix} \tag{A21}$$

$$\{Y'\} = \begin{Bmatrix} -b_{,1} \\ (E_c\nu_c/\gamma_c)-F_{1,1} \\ b_{,1}-b_{,2} \\ F_{1,1}-F_{1,2} \\ \\ b_{,i}-b_{,i+1} \\ F_{1,i}-F_{1,i+1} \\ \text{-----------} \\ F_{1,n-1}-F_{1,n} \\ -p_e+F_{1,n} \end{Bmatrix} \tag{A22}$$

with $i = 1, n$ and letting

$$R_{1,i} = r_i^{k_{1,i}} \tag{A23}$$

$$R_{2,i} = r_i^{k_{2,i}} \tag{A24}$$

$$S_{1,i} = -r_i^{k_{1,i}-1} \tag{A25}$$

$$S_{2,i} = -r_i^{k_{2,i}-1} \tag{A26}$$

non-vanishing elements of $(2n+1)\times(2n+1)$ matrix $[M]$ are given by

$$M_{11} = 1, \qquad M_{12} = S_{1,1}, \qquad M_{13} = S_{2,1}$$

$$M_{21} = E_c/\gamma_c, \qquad M_{22} = C_{1,1}S_{1,1}, \qquad M_{23} = D_{1,1}S_{2,1}$$

$$M_{j,k} = R_{1,i}, \qquad M_{j,k+1} = R_{2,i}$$

$$M_{j,k+2} = S_{1,i+1}, \qquad M_{j,k+3} = S_{2,i+1}$$

$$M_{j+1,k} = C_{1,i}R_{1,i}, \qquad M_{j+1,k+1} = D_{1,i}R_{2,i}$$

$$M_{j+1,k+2} = C_{1,i+1}S_{1,i+1}, \qquad M_{j+1,k+3} = D_{1,i+1}S_{2,i+1}$$

for $j = 2i+1, \quad k = 2i, \quad i = 1,\ldots,n-1$

$$M_{2n+1,2n} = C_{1,n}R_{1,n}, \qquad M_{2n+1,2n+1} = D_{1,n}R_{2,n}. \tag{A27}$$

4. *Global stiffness matrix*

Consider one isolated cylinder under internal pressure p_{in} and external pressure p_{ex}. Boundary conditions on normal stress at internal radius r_{in} and external radius r_{ex} are

$$q_1 C_1 r_{in}^{k_1} + q_2 D_1 r_{in}^{k_2} + E_1 \phi r_{in} + F_1 \varepsilon = -p_{in} \tag{A28}$$

$$q_1 C_1 r_{ex}^{k_1} + q_2 D_1 r_{ex}^{k_2} + E_1 \phi r_{ex} + F_1 \varepsilon = -p_{ex} \tag{A29}$$

or, in matrix form

$$[\bar{M}]\{q\} = \phi\{\bar{X}'\} + \omega\{\bar{W}\} + \{\bar{Y}'\} \tag{A30}$$

where

$$[\bar{M}] = \begin{bmatrix} C_1 r_{in}^{k_1} & D_1 r_{in}^{k_2} \\ C_1 r_{ex}^{k_1} & D_1 r_{ex}^{k_2} \end{bmatrix} \tag{A31}$$

$$\{\bar{X}'\}^{T} = \{-E_1 r_{in} \quad -E_1 r_{ex}\} \tag{A32}$$

$$\{\bar{W}'\}^{T} = \{-F_1 \quad F_1\} \tag{A33}$$

$$\{\bar{Y}'\}^{T} = \{F_1 - p_{in} \quad F_1 - p_{ex}\}. \tag{A34}$$

Multiplying by $[\bar{M}]^{-1}$, one obtains

$$\{q\} = \phi\{\bar{X}\} + \omega\{\bar{W}\} + \{\bar{Y}\} \tag{A35}$$

that is

$$q_1 = \phi \bar{X}_1 + \omega \bar{W}_1 + \bar{Y}_1 \tag{A36}$$

$$q_2 = \phi \bar{X}_2 + \omega \bar{W}_2 + \bar{Y}_2. \tag{A37}$$

Now, calculating the tensile load $\bar{N}$ on the cylinder, from eqn (44), one obtains

$$\bar{N} = \omega \bar{N}_1 + \phi \bar{N}_2 + \bar{N}_3 \tag{A38}$$

where $\bar{N}_1$, $\bar{N}_2$, $\bar{N}_3$ are given by

$$\bar{N}_1 = 2\pi \left[\bar{W}_1 C_3 \frac{r_{ex}^{k_1+2} - r_{in}^{k_1+2}}{k_1+2} + \bar{W}_2 D_3 \frac{r_{ex}^{k_2+2} - r_{in}^{k_2+2}}{k_2+2} + F_3 \frac{r_{ex}^2 - r_{in}^2}{2} \right] \tag{A39}$$

$$\bar{N}_2 = 2\pi \left[\bar{X}_1 C_3 \frac{r_{ex}^{k_1+2} - r_{in}^{k_1+2}}{k_1+2} + \bar{X}_2 D_3 \frac{r_{ex}^{k_2+2} - r_{in}^{k_2+2}}{k_2+2} + E_3 \frac{r_{ex}^3 - r_{in}^3}{3} \right] \tag{A40}$$

$$\bar{N}_3 = 2\pi \left[\bar{Y}_1 C_3 \frac{r_{ex}^{k_1+2} - r_{in}^{k_1+2}}{k_1+2} + \bar{Y}_2 D_3 \frac{r_{ex}^{k_2+2} - r_{in}^{k_2+2}}{k_2+2} - F_3 \frac{r_{ex}^2 - r_{in}^2}{2} \right]. \tag{A41}$$

In the same fashion, the twisting moment $\bar{M}$ on the cylinder is obtained from eqn (46), yielding

$$\bar{M} = \omega \bar{M}_1 + \phi \bar{M}_2 + \bar{M}_3 \tag{A42}$$

where $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$ are given by

$$\bar{M}_1 = 2\pi \left[\bar{W}_1 C_4 \frac{r_{ex}^{k_1+3} - r_{in}^{k_1+3}}{k_1+3} + \bar{W}_2 D_4 \frac{r_{ex}^{k_2+3} - r_{in}^{k_2+3}}{k_2+3} + F_4 \frac{r_{ex}^3 - r_{in}^3}{3} \right] \tag{A43}$$

$$\bar{M}_2 = 2\pi \left[\bar{X}_1 C_4 \frac{r_{ex}^{k_1+3} - r_{in}^{k_1+3}}{k_1+3} + \bar{X}_2 D_4 \frac{r_{ex}^{k_2+3} - r_{in}^{k_1+3}}{k_2+3} + E_4 \frac{r_{ex}^4 - r_{in}^4}{4} \right] \tag{A44}$$

$$\bar{M}_3 = 2\pi \left[\bar{Y}_1 C_4 \frac{r_{ex}^{k_1+3} - r_{in}^{k_1+3}}{k_1+3} + \bar{Y}_2 D_4 \frac{r_{ex}^{k_1+3} - r_{in}^{k_2+3}}{k_2+3} - F_4 \frac{r_{ex}^3 - r_{in}^3}{3} \right]. \tag{A45}$$

Now for a solid cylinder under axial load and radial pressure $p_{ex,0}$, with an elastic, isotropic material, the foregoing equations reduce to

$$\bar{N}_{,0} = \omega \bar{N}_{1,0} + \bar{N}_{3,0} \tag{A46}$$

and

$$\bar{M}_{,0} = \phi \bar{M}_{2,0} \tag{A47}$$

with

$$\bar{N}_{1,0} = A_c E_c \tag{A48}$$

$$\bar{N}_{3,0} = -A_c E_c \left(1 + 2\nu_c \frac{p_{ex,0}}{E_c} \right) \tag{A49}$$

$$\bar{M}_{2,0} = G_0 J_c \tag{A50}$$

A_c and J_c being the core cross-section area and polar inertia, respectively.

Thus, finally, for a system of n cylinders with an isotropic core, global parameters N_j and M_j ($j = 1, 2, 3$) from eqns (45) and (47) are given by

$$N_j = \sum_{k=0}^{n} \bar{N}_{j,k} \tag{A51}$$

$$M_j = \sum_{k=0}^{n} \bar{M}_{j,k} \tag{A52}$$

where $\bar{N}_{j,k}$ and $\bar{M}_{j,k}$ are obtained from eqns (A39) to (A50) by indexing all quantities for cylinder k.